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Complex Mean-Value Interpolation and Approximation of Holomorphic Functions

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We show that complex mean-value interpolation, a generalization of Lagrange–Hermite interpolation, may be defined in any domain that is $\mathbb{C}$-convex, whereas the original definition required ordinary, real convexity. We also show that $\mathbb{C}$-convex domains are the natural ones in which to perform mean-value interpolation, in the sense that any Runge domain which admits mean-value interpolation must in fact be $\mathbb{C}$-convex. Finally, we obtain an integral formula for the error and give some applications concerning approximation of holomorphic functions.

1. INTRODUCTION

The simplest instance of mean-value interpolation is ordinary Lagrange–Hermite interpolation in $\mathbb{R}$: given a sufficiently smooth function f and a sequence of points $p=(p_0, ..., p_k)$, possibly coincident, there is a unique polynomial $L_p f$ of degree at most k interpolating f at the points p (including derivatives in the case of multiple points).

In one variable, the general mean-value interpolation operators, L_p^m, can be defined in terms of the Lagrange–Hermite operator L_p as follows. For any integer m such that $0 \leqslant m \leqslant k$ we let $D^{-m}f$ be any function such that $D^m(D^{-m}f)=f$. Then

$$L_p^m f = D^m L_p D^{-m} f.$$

This polynomial of degree $k-m$ has the property of matching certain canonical mean-values of the function f.

These one variable methods were extended to $\mathbb{R}^n$ by Goodman [14] and their extensions are sometimes referred to as the scale of mean-value interpolations. As special cases there appear the analogues in $\mathbb{R}^n$ of

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ordinary Lagrange–Hermite interpolation, originally found by Kergin [18] and Hakopian [15].

In this paper we turn to $\mathbb{C}^n$ and show that, whereas the real mean-value interpolation operators are confined to functions defined on convex domains, their complex analogues can be substantially extended. To be precise, we prove that any $\mathbb{C}$-convex domain gives rise to a unique scale of mean-value interpolation operators. And we also prove a converse, namely that any Runge domain which admits mean-value interpolation must in fact be $\mathbb{C}$-convex. Hence, $\mathbb{C}$-convex domains are the natural ones in which to perform mean-value interpolation. The special case of complex Kergin interpolation was treated by Andersson and Passare in [4] and [5]. For the other cases, no complex results have previously been given.

We also give an integral formula for the error in complex mean-value interpolation, generalizing a classical formula of Hermite. Using the error formula, we obtain results about the convergence of mean-value interpolating polynomials to holomorphic functions. Such convergence theorems are known for complex Kergin interpolation, but again in the other cases nothing has previously been proved.

The paper is set out in the following way. Sections 3 and 4 contain some background material and a discussion of mean-value interpolation in $\mathbb{R}$ and $\mathbb{R}^n$. In Sections 5 and 6 the basic tools for introducing complex mean-value interpolation are presented. The main results, Theorem 7.3 and Theorem 7.6, are stated and proven in Section 7. In Section 8 we obtain an integral formula for the error. This formula is used in Sections 9 and 10 to approximate holomorphic functions. There is a close connection between complex convexity and the Fantappiè transform. This gives rise to a different way of looking at complex mean-value interpolation, discussed in Section 11. Finally, Section 12 deals with mean-value interpolation from the point of view of numerical analysis.

2. NOTATION

The space of n-variate complex polynomials of degree k will be denoted $\Pi_k(\mathbb{C}^n)$. Given a sequence of points $p=(p_0, ..., p_k)$ the notation $p' \subset p$ means that p' is a subsequence of p and $p \setminus p'$ denotes the complementary subsequence. The subsequence consisting of the first $j+1$ terms of p is denoted p^j. The cardinality of the sequence p will be written $\#p$.

For a one-variable function g we write $D^j g$ for the jth derivative of g, and for a multivariate function f and multi-index α we let

$$D^\alpha f = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} f.$$

The partial derivative of a function f in the directions p is denoted $D_p f = D_{p_0} \cdots D_{p_k} f$. Thus, for example, with $p(p_0, ..., p_6)$, we have that

$$D_{z - p \setminus p^4,\, z - p_2} f = D_{z - p_5} D_{z - p_6} D_{z - p_2} f.$$

To distinguish between interpolation operators in one and several variables we use script letters for the latter.

For a domain $\Omega \subset \mathbb{C}^n$ we let $\mathcal{O}(\Omega)$ denote the space of functions holomorphic on Ω.

3. THE REAL SIMPLEX FUNCTIONAL AND MEAN-VALUE INTERPOLATION IN $\mathbb{R}$

Recall that, in one variable, the Newton divided differences of a function f at the points $p = (p_0, ..., p_k)$ are defined recursively by

$$[p_i]\, f = f(p_i),$$

$$[p_0, ..., p_j]\, f = \frac{[p_1, ..., p_j]\, f - [p_0, ..., p_{j-1}]\, f}{p_j - p_0}.$$

It is well known that the divided differences are symmetric functions in their arguments and that they have limiting confluent forms as points (which from the outset are supposed to be distinct) approach each other. For example, if $p = (p_0, ..., p_0)$ where p_0 is repeated $j+1$ times, then $[p_0, ..., p_0]\, f = f^{(j)}(p_0)/j!$.

Now the Lagrange–Hermite polynomial $L_p f$ interpolating a function f at the points $p = (p_0, ..., p_k)$, including derivatives up to the corresponding order in the case of multiple points, can be written

$$L_p f(x) = f(p_0) + (x - p_0)[p_0, p_1]\, f + ...$$
$$+ (x - p_0)(x - p_1) \cdots (x - p_{k-1})[p_0, ..., p_k]\, f.$$

Note that this formula contains, as special cases, both the Lagrange polynomial (if the points are distinct) and the Taylor polynomial (if one point is repeated $k+1$ times).

Closely linked to the divided differences in one variable, and crucial to the construction of mean-value interpolation maps, is the following functional, called the Simplex functional.

DEFINITION 3.1. Let $p=(p_0, ..., p_j)$ be a sequence of points in $\mathbb{R}^n$. The Simplex functional with respect to p is defined by

$$f \mapsto \int_{[p]} f = \int_{\Delta^j} f(p_0 + s_1(p_1 - p_0) + \cdots + s_j(p_j - p_0))\, ds_1 \cdots ds_j,$$

where $\Delta^j = \{(s_1, ..., s_j);\ s_i \geqslant 0, s_1 + \cdots + s_j \leqslant 1\}$ is the standard j-simplex.

In $\mathbb{R}$, the Simplex functional is related to the divided differences by the classical Hermite–Genocchi formula:

$$[p_0, ..., p_j]\, f = \int_{[p_0, ..., p_j]} f^{(j)}.$$

Thus, the Lagrange–Hermite polynomial can also be written in the following form, suitable for generalization to several variables:

$$\begin{aligned} L_p f(x) = f(p_0) + (x - p_0) \int_{[p_0, p_1]} f' \\ + \cdots + (x-p_0)(x-p_1) \cdots (x - p_{k-1}) \int_{[p_0, ..., p_k]} f^{(k)}. \end{aligned}$$

From the Hermite–Genocchi formula it is apparent that a polynomial Q interpolates f at the points p, including derivatives in the case of multiple points, if and only if

$$\int_{[p_0, ..., p_j]} D^j Q = \int_{[p_0, ..., p_j]} D^j f, \qquad j = 0, ..., k,$$

and so the Lagrange–Hermite polynomial is the unique polynomial of degree k matching these mean-values of the function f.

It is equally natural to interpolate other mean-values. In the general case we have the following well known result (see e.g. [14]):

THEOREM 3.2. *Let $p=(p_0, ..., p_k)$ be a sequence of points in $\mathbb{R}$ and let m be any integer such that $0 \leqslant m \leqslant k$. Then, for any function $f \in C^{k-m}(\mathbb{R})$, there exists a unique polynomial $L_p^m f$ of degree $k-m$ such that*

$$\int_{[p_0, ..., p_{j+m}]} D^j (f - L_p^m f) = 0, \qquad j = 0, ..., k-m.$$

Moreover, the polynomial $L_p^m f$ *is given by*

$$L_p^m f(x) = m! \sum_{r=m}^{k} \sum_{\substack{p' \subset p^{r-1} \\ \# p' = r-m}} \prod_{p_j \in p'} (x - p_j) \int_{[p_0, \ldots, p_r]} D^{r-m} f.$$

Proof. Let $D^{-m}f$ be any function such that $D^m(D^{-m}f) = f$ and let L_p be the ordinary Lagrange–Hermite interpolation operator at the points p. Then we claim that the desired polynomial is

$$D^m(L_p(D^{-m}f)),$$

i.e., the polynomial obtained by first taking an mth primitive function of f, then finding the Lagrange–Hermite polynomial of this function at the points p, and then finally taking the mth derivative of this polynomial. This procedure clearly gives a well defined polynomial of the desired degree.

It is immediate from the Hermite–Genocchi formula that for $j = 0, \ldots, k - m$,

$$\int_{[p_0, \ldots, p_{j+m}]} D^j f = [p_0, \ldots, p_{j+m}]\, D^{-m} f,$$

and

$$\int_{[p_0, \ldots, p_{j+m}]} D^j(D^m(L_p(D^{-m}f))) = [p_0, \ldots, p_{j+m}]\, L_p(D^{-m}f).$$

Since the Lagrange–Hermite polynomial interpolates function values at the points in question, these divided differences are equal.

For the uniqueness part, suppose there are two polynomials, Q and R, meeting the requirements. Then

$$\int_{[p_0, \ldots, p_{j+m}]} D^j(Q - R) = 0, \qquad j = 0, \ldots, k - m.$$

Taking $j = k - m$ in this formula, we see that the highest order coefficient of Q and R, i.e., the coefficient of x^{k-m}, must be equal. Having proved this, we now take $j = k - m - 1$ in the formula and conclude that also the coefficients of x^{k-m-1} are equal. Continuing this process down to $j = 0$ we see that $Q = R$, and the uniqueness is established.

To see that the polynomial can be written in Newton form as claimed, note first that it is obviously true for $m = 0$, then use the formula $L_p^m f = D^m L_p D^{-m} f$. ∎

Remark 3.3. For $m=0$ we of course recover the ordinary Lagrange–Hermite polynomial. For $m=1$ we obtain what has been called an area matching map, since in this case, if the points are distinct, the interpolation conditions are equivalent to

$$\int_{p_j}^{p_{j+1}} (f - L_p^m f)(x)\, dx = 0, \qquad j = 0, ..., k-1.$$

For arbitrary m the interpolation described here is natural and has been used, for example by de Boor [10], to bound spline interpolation.

Remark 3.4. The interpolation conditions are usually stated slightly differently. The polynomial $L_p^m f$ is then required to satisfy

$$\int_{[p']} D^j (f - L_p^m f) = 0,$$

for all subsequences $p' \subset p$ such that $\# p' \geqslant m+1$ and $j = \# p' - m - 1$.

We see from the proof above that the conditions given in Theorem 3.2 are sufficient to determine the polynomial (see also Remark 4.5 where the multivariate case is discussed).

Yet another equivalent way to write the interpolation conditions is to single out the lowest order conditions, and require that

$$\int_{[p']} (f - L_p^m f) = 0,$$

for all subsequences $p' \subset p$ such that $\# p' = m+1$. It is straightforward calculation to prove that these lowest order conditions actually imply the higher order conditions involving derivatives. Note that for $m=0$ this is just point evaluation. For $m=1$ this is the formulation used in the preceding remark.

Remark 3.5. A result analogous to Theorem 3.2 holds in $\mathbb{C}$: If $p = (p_0, ..., p_k)$ is a sequence of points from $\mathbb{C}$ and m is an integer, $0 \leqslant m \leqslant k$, then for any function $f \in \mathcal{O}(\mathbb{C})$ there exists a unique polynomial $L_p^m f$ of degree $k-m$ such that

$$\int_{[p_0, ..., p_{j+m}]} D^j (f - L_p^m f) = 0, \qquad j = 0, ..., k-m.$$

The proof of this is no different from the proof in the real case, it is just a question of interpreting the entities involved properly.

However, the complex result can be extended. To begin with, it is easy to see that it is enough for the function f to be holomorphic on the convex hull of the points p. But we can go further. If we start by fixing a simply connected domain Ω containing the points p, then for every function f holomorphic in Ω there is a unique polynomial $L_p^m f$ with the required properties (cf. Theorem 7.3).

4. MEAN-VALUE INTERPOLATION IN $\mathbb{R}^n$

In order to generalize mean-value interpolation to several variables, we need to make precise in what way the one variable operator and the several variable operator should be connected. This is done in the following definition, in which we denote the linear form on $\mathbb{R}^n$, induced by scalar product with $\psi \in \mathbb{R}^n$, by Ψ, i.e., for $x \in \mathbb{R}^n$, $\Psi(x) = \sum_{j=1}^n \psi_j x_j$.

Definition 4.1. Let there to each sequence of points $p = (p_0, ..., p_k)$ in $\mathbb{R}$ be associated a continuous linear map M_p: $C^s(\mathbb{R}) \to C(\mathbb{R})$. A continuous linear map $\mathscr{M}_p$: $C^s(\mathbb{R}^n) \to C(\mathbb{R}^n)$ is said to be the lift of M to p in $\mathbb{R}^n$ if its satisfies

$$\mathscr{M}_p(g \circ \Psi) = (M_{\Psi(p)}\, g) \circ \Psi, \qquad \forall \psi \in \mathbb{R}^n, \quad \forall g \in C^s(\mathbb{R}).$$

Remark 4.2. What this amounts to is precisely that the operator $\mathscr{M}_p$ is required to be invariant under affine mappings. Observe, for example, that this is the relationship between the Taylor operator in one and several variables.

The mean-value interpolation maps were lifted in the above sense to $\mathbb{R}^n$ by Goodman [14], who proves a weaker version of the following result:

Theorem 4.3. *Let $p = (p_0, ..., p_k)$ be a sequence of points (possibly coincident) in $\mathbb{R}^n$ and let m be an integer such that $0 \leqslant m \leqslant k$. Then for any function $f \in C^{k-m}(\mathbb{R}^n)$ there is a unique polynomial $\mathscr{L}_p^m f \in \Pi_{k-m}(\mathbb{R}^n)$ such that*

$$\int_{[p_0, ..., p_{j+m}]} D^\alpha (f - \mathscr{L}_p^m f) = 0$$

for all $j = 0, ..., k-m$, and all multi-indices α with $|\alpha| = j$.

Moreover, the polynomial $\mathscr{L}_p^m f$ is given by

$$\mathscr{L}_p^m f(x) = m! \sum_{r=m}^{k} \sum_{\substack{p' \subset p^{r-1} \\ \# p' = r-m}} \int_{[p_0, ..., p_r]} D_{x-p'} f.$$

Proof. Note that an operator satisfying the conditions of the theorem has to be continuous, so it is sufficient to prove the theorem for so called ridge functions or plane waves. These are functions of the type $f = g \circ \Psi$, where g is a one variable function and Ψ is the linear form induced by scalar product with $\psi \in \mathbb{R}^n$. For any such function, define a polynomial q in $\mathbb{R}^n$ by

$$q = (D^m(L_{\Psi(p)}(D^{-m}g))) \circ \Psi.$$

It is a straightforward calculation, using the Hermite–Genocchi formula, to see that this polynomial indeed satisfies the condition in the theorem, and that it in fact can be written in Newton form as claimed (see [14] for the details).

For the uniqueness part, suppose there are two polynomials, Q and R, meeting the requirements. Then

$$\int_{[p_0, \ldots, p_{j+m}]} D^\alpha(Q - R) = 0$$

for all $j = 0, \ldots, k-m$ and all multi-indices α with $|\alpha| = j$. Taking $j = k-m$ in this formula we see that for any multi-index α with $|\alpha| = k-m$, the x^α coefficient of Q and R must be equal (since $D^\alpha(Q-R)$ is a constant having zero integral over a set of positive measure). Having proved this, we know that $Q - R$ is a polynomial of degree at most $k-m-1$, and we proceed by taking $j = k-m-1$ in the formula and conclude that for any multi-index β of length $k-m-1$ the x^β-coefficients of Q and R are equal. Continuing this process down to the case $j = 0$, we get that $Q = R$ as desired. ∎

Remark 4.4. Several special cases are worth pointing out. For $m = 0$ we recover the Kergin map. The explicit formula for the interpolating polynomial was found by Micchelli and Milman [19] and [20]. In Newton form, the Kergin polynomial is

$$\mathscr{L}_p^0 f(x) = f(p_0) + \int_{[p_0, p_1]} D_{x-p_0} f$$
$$+ \cdots + \int_{[p_0, \ldots, p_k]} D_{x-p_0} D_{x-p_1} \cdots D_{x-p_{k-1}} f.$$

For $m = 1$ we get a map studied by Cavaretta, Micchelli, and Sharma [12], the lifting of the area matching map discussed in Remark 3.3.

For $m = n-1$ we obtain a map introduced by Hakopian [15].

Remark 4.5. The interpolation conditions are usually stated differently. Actually, in [14] the polynomial $\mathscr{L}_p^m f$ is required to satisfy

$$\int_{[p']} D^\alpha(f - \mathscr{L}_p^m f) = 0$$

for all subsequences $p' \subset p$ such that $\# p' \geqslant m+1$ and for all multi-indices α such that $|\alpha| = \# p' - m - 1$.

Goodman points out in [14] that these linear functional are not linearly independent, but claims that it is very complicated to give a linearly independent collection of them explicitly. The conditions of Theorem 4.3 give an answer to this problem, since their number is equal to the dimension of the interpolating polynomial space and it is proved above that they in fact determine the polynomial.

Remark 4.6. Observe that, given a sequence $p = (p_0, ..., p_k)$ of points in $\mathbb{C}^n$, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and to any entire function f associate the polynomial $\mathscr{L}_p^m \operatorname{Re} f + i\mathscr{L}_p^m \operatorname{Im} f$. This polynomial is holomorphic. By Theorem 4.3 it is the unique polynomial of degree at most $k-m$ satisfying the interpolation conditions. In this way mean-value interpolation can be generalized to $\mathbb{C}^n$. However, using the complex convexity notions introduced in Section 5, we will be able to define complex mean-value interpolation in a much more general setting in Section 7.

5. COMPLEX CONVEXITY

It turns out that the complex analogues of the real mean-value interpolation maps are closely linked with complex convexity. There is a quite extensive theory of complex notions of convexity, developed over recent years. Here we just include a brief presentation of some basic concepts and results of particular importance to us. For a detailed and unified treatment of this theory, we refer the reader to the work by Andersson, *et al.* [6], where also further references can be found.

There are several equivalent characterizations of the convex domains in $\mathbb{R}^n$. For example, it is well known that an open, connected set $\Omega \subset \mathbb{R}^n$ is convex if and only if any one of the following conditions holds:

(1) The intersection of Ω with an arbitrary real line is contractible (or empty).

(2) The complement of Ω is a union of hyperplanes.

(3) Through every boundary point of Ω passes a hyperplane that does not intersect Ω.

Obviously, domains in $\mathbb{C}^n$ can be convex in this sense, but there are also other analogous purely complex concepts. These arise when the conditions above are generalized to the complex setting.

The first of these complex convexity concepts is defined in analogy with (1).

DEFINITION 5.1. A domain $\Omega \subset \mathbb{C}^n$ is said to be $\mathbb{C}$-convex if its intersection with an arbitrary complex line is contractible (or empty).

It is clear that convexity implies $\mathbb{C}$-convexity. However, the converse is not true. In dimension one this is obvious, since in this case a domain is $\mathbb{C}$-convex if and only if it is simply connected. But it is, in fact, easy to construct bounded $\mathbb{C}$-convex domains in any dimension that are not convex. See Examples 2.2.5, 2.2.6, and 2.4.10 of [6]. It is worth pointing out that the notion of $\mathbb{C}$-convexity can be defined also for compact sets, and in this case a compact $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ is convex if and only if it is $\mathbb{C}$-convex.

The generalization of condition (2) gives rise to a concept slightly weaker than $\mathbb{C}$-convexity.

DEFINITION 5.2. A domain $\Omega \subset \mathbb{C}^n$ is said to be lineally convex if its complement is a union of complex hyperplanes.

It is proved in Theorem 2.3.7 of [6] that $\mathbb{C}$-convexity implies lineal convexity. The converse does not hold in general, as follows from the fact that the intersection of $\mathbb{C}$-convex sets need not be $\mathbb{C}$-convex. However, if some boundary regularity is assumed, the two concepts are equivalent.

THEOREM 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^1 boundary. Then Ω is $\mathbb{C}$-convex if and only if it is lineally convex.*

Proof. See Theorem 2.3.7 and Corollary 2.4.5 in [6]. ▌

It is obvious that any lineally convex open set in $\mathbb{C}^n$ is pseudo-convex. Hence in the complex case there is a scale of notions of convexity, ranging from pseudo-convexity via lineal convexity and $\mathbb{C}$-convexity to ordinary convexity.

The following property of the $\mathbb{C}$-convex domains will be of importance to us.

THEOREM 5.4. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain. Then Ω is a Runge domain.*

Proof. See Proposition 2.1.9 in [6]. ▌

We point out that this means that if Ω is a $\mathbb{C}$-convex domain, then any function holomorphic in Ω can be approximated uniformly on compact subsets by entire functions, and hence by polynomials.

The generalization of condition (3) induces yet another complex notion of convexity.

DEFINITION 5.5. An open set $\Omega \subset \mathbb{C}^n$ is said to be weakly lineally convex if through every boundary point of Ω of there passes a complex hyperplane that does not intersect Ω.

We will not use the concept of weak lineal convexity here. We just mention that weakly lineally convex open sets are precisely those sets that can show up as connected components of lineally convex open sets.

6. THE COMPLEX SIMPLEX FUNCTIONAL

Our aim in this section is to define the complex version of the Simplex functional of Section 3. This was originally done in [4], and the construction is as follows.

Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain and $p = (p_0, ..., p_k)$ a sequence of points in Ω. Denote the standard j-simplex in $\mathbb{R}^j$ by Δ^j, its vertices by $v_0, ..., v_j$, and for each $j \leqslant k$ let Ω^j be the intersection of Ω with the complex affine space spanned by $p_0, ..., p_j$. Also, let $\omega^j \subset \mathbb{C}^j$ be the preimage of Ω^j under the complex affine mapping $\mathbb{C}^j \to \mathbb{C}^n$ taking each v_i to p_i (using, of course, the canonical inclusion $\mathbb{R}^j \subset \mathbb{C}^j$). It turns out that ω^j is again $\mathbb{C}$-convex. Finally, introduce singular chains $\gamma^j \colon \Delta^j \to \omega^j$ mapping every face of Δ^j into the complex $(j-1)$-plane which it spans. This is possible by $\mathbb{C}$-convexity, and it follows that each v_i is fixed.

DEFINITION 6.1. With the notation introduced above, the complex Simplex functional is defined to be

$$f \mapsto \int_{[p_0, ..., p_j]} f = \int_{\gamma^j} f(p_0 + \lambda_1(p_1 - p_0) + \cdots + \lambda_j(p_j - p_0))\, d\lambda_1 \wedge \cdots \wedge d\lambda_j.$$

Observe that the complex Simplex functional depends on the domain Ω; see also Remark 7.5.

We now list some properties of the complex Simplex functional.

PROPOSITION 6.2. *For $p = (p_0, ..., p_k) \subset \Omega$, the complex Simplex functional*

$$\int_{[p_0, ..., p_j]} : \mathcal{O}(\Omega) \to \mathbb{C}$$

defined above is independent of the particular choice of chain γ^j *in* ω^j. *Moreover,*

(i) *it is independent of the order of the points* p,

(ii) *it is invariant under complex affine mappings, i.e., if* $\Psi: \mathbb{C}^j \to \mathbb{C}^l$ *is any such map, then*

$$\int_{[p_0, ..., p_j]} f \circ \Psi = \int_{[\Psi(p_0), ..., \Psi(p_j)]} f,$$

(iii) *if* $f \in \mathcal{O}(\Omega)$, *then the map*

$$(p_0, ..., p_j) \mapsto \int_{[p_0, ..., p_j]} f$$

is holomorphic.

Proof. For the proof of the independence of the choice of chain, see Proposition 8 in [2]. Properties (i)—(iii) are quite immediate consequences of the definition. ∎

7. COMPLEX MEAN-VALUE INTERPOLATION

We are now in a position to define complex mean-value interpolation and prove our main results, Theorems 7.3 and 7.6 below.

DEFINITION 7.1. Given a $\mathbb{C}$-convex domain $\Omega \subset \mathbb{C}^n$, let $f: \Omega \to \mathbb{C}$ be a holomorphic function, $p = (p_0, ..., p_k)$ a sequence of points in Ω, and m an integer such that $0 \leqslant m \leqslant k$. The mean-value interpolating polynomial $\mathcal{L}^m_{p, \Omega} f$ of f with respect to Ω and the points p is

$$\begin{aligned}
&\mathcal{L}^m_{p, \Omega} f(z) \\
&\quad = m! \sum_{r=m}^{k} \sum_{\substack{p' \subset p^{r-1} \\ \# p' = r-m}} \int_{[p^r]} D_{z-p'} f \\
&\quad = m! \left[\int_{[p_0, ..., p_m]} f + \sum_{j=0}^{m} \int_{[p_0, ..., p_{m+1}]} D_{z-p_j} f \right. \\
&\qquad \left. + \cdots + \sum_{0 \leqslant j_1 < j_2 < \cdots < j_{k-m} \leqslant k-1} \int_{[p_0, ..., p_k]} D_{z-p_{j_1}} D_{z-p_{j_2}} \cdots D_{z-p_{j_{k-m}}} f \right].
\end{aligned}$$

Remark 7.2. If Ω is convex, or if $f \in \mathcal{O}(\Omega)$ can be continued to a function holomorphic on the convex hull of the points p, we can let the γ^j in the definition of the Simplex functional be identity mappings.

THEOREM 7.3. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain, $p = (p_0, ..., p_k)$ a sequence of points in Ω, and m an integer, $0 \leqslant m \leqslant k$. The mean-value interpolation operator $f \mapsto \mathscr{L}^m_{p, \Omega} f$ is the unique linear operator $\mathcal{O}(\Omega) \to \Pi_{k-m}(\mathbb{C}^n)$ satisfying*

$$\int_{[p_0, ..., p_{j+m}]} D^\alpha (f - \mathscr{L}^m_{p, \Omega} f) = 0, \tag{$*$}$$

for any $j = 0, ..., k - m$ and any multi-index α with $|\alpha| = j$.

Moreover,

(i) *it is independent of the ordering of the points p_j,*

(ii) *it is continuous (in the usual topologies of $\mathcal{O}(\Omega)$ and $\Pi_{k-m}(\mathbb{C}^n)$),*

(iii) *it is holomorphic as a function of p,*

(iv) *it is invariant under complex affine mappings, i.e., $\mathscr{L}^m_{p, \Omega}(f \circ \Psi) = (\mathscr{L}^m_{\Psi(p), \Psi(\Omega)} f) \circ \Psi$ for any such mapping $\Psi\colon \mathbb{C}^n \to \mathbb{C}^l$,*

(v) *it is a projection onto $\Pi_{k-m}(\mathbb{C}^n)$,*

(vi) *it is invariant under restriction:*

$$p \subset \Omega' \subset \Omega \Rightarrow \mathscr{L}^m_{p, \Omega'} f(z) = \mathscr{L}^m_{p, \Omega} f(z), \qquad \forall z \in \Omega',$$

(vii) *it reduces to the Taylor operator, with interpolant of degree $k - m$, in case $p = (p_0, ..., p_0)$ consists of one point repeated $k + 1$ times,*

(viii) *it is associative: $p \subset q \Rightarrow \mathscr{L}^m_{p, \Omega} \mathscr{L}^m_{q, \Omega} f = \mathscr{L}^m_{p, \Omega} f$.*

Proof. First we observe (cf. Remark 4.6) that mean-value interpolation carries over to the complex case in much the same way as ordinary Lagrange–Hermite interpolation. That is, for $f \in \mathcal{O}(\mathbb{C}^n)$ we can, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, use the ordinary, real $\mathscr{L}^m_p$ operator and put $\mathscr{L}^m_p f = \mathscr{L}^m_p(\operatorname{Re} f) + i\mathscr{L}^m_p(\operatorname{Im} f)$. This complex polynomial is holomorphic and it is precisely the polynomial we get using our Definition 7.1, letting the γ^j in the Simplex functionals be identity mappings.

Thus the results of [14] are available to us in the case of entire functions. This will be of great importance throughout this proof.

The properties (ii) and (iii) follow immediately from the definition, and so do properties (i) and (iv) using Proposition 6.2.

Now Ω is Runge by Theorem 5.4, so $\mathcal{O}(\mathbb{C}^n)$ is dense in $\mathcal{O}(\Omega)$, and ($*$) follows by the continuity of the mean-value interpolation operator and by the same property of ordinary mean-value interpolation.

The property (v) will follow immediately, once we have established the uniqueness assertion.

The invariance under restrictions follows from the uniqueness for polynomials and the fact that $\mathcal{O}(\Omega) \subset \mathcal{O}(\Omega')$ is an injection.

The properties (vii) and (viii) are easily verified from the definition, in view of ($*$).

It remains to show uniqueness. Suppose that there are two polynomials, Q and R, meeting the requirements. Then

$$\int_{[p_0, ..., p_{j+m}]} D^{\alpha}(Q-R) = 0$$

for any $j = 0, ..., k-m$ and any multi-index α with $|\alpha| = j$. Taking $j = k-m$ in this formula we obtain that for any multi-index α with $|\alpha| = k-m$, the z^{α}-coefficient of Q and R must be equal. Having proved this, we proceed by taking $j = k-m-1$ and using the formula to conclude that Q and R have equal z^{β} coefficient for all multi-indices β with $|\beta| = k-m-1$. Continuing this process down to the case $j = 0$ we get that $Q = R$, and the uniqueness is established. ▮

Remark 7.4. For $m = 0$ we recover the complex version of Kergin interpolation studied by Andersson and Passare in [4] and [5]. This is the only instance of mean-value interpolation previously studied in the complex case. The interpolating polynomial is given by

$$\mathcal{L}^0_p f(z) = f(p_0) + \int_{[p_0, p_1]} D_{z-p_0} f$$
$$+ \cdots + \int_{[p_0, ..., p_k]} D_{z-p_0} D_{z-p_1} \cdots D_{z-p_{k-1}} f.$$

For $m = 1$ we obtain the complex extension of the interpolation operator studied in the real cases by Cavaretta, *et al.* [12], the lifting of the area matching map.

For $m = k$ the interpolating polynomial is a constant, namely

$$\mathcal{L}^k_{p,\Omega} f = k! \int_{[p_0, ..., p_k]} f.$$

For $m = n-1$ we get the complex extension of the interpolation operator studied in the real case by Hakopian [15].

Remark 7.5. For entire functions it follows from (vi) that the mean-value interpolating polynomial is independent of the particular choice of domain Ω containing the points p, and so in this case we may simply write

$\mathscr{L}^m_p f$. However, in general, if Ω and Ω' are $\mathbb{C}$-convex domains such that $f \in \mathcal{O}(\Omega \cup \Omega')$ and $p \subset \Omega \cap \Omega'$, it is not necessarily true that $\mathscr{L}^m_{p,\Omega} f(z) = \mathscr{L}^m_{p,\Omega'} f(z)$, as was shown in [5]. Take for instance $k=1$, $m=0$, and

$$\Omega^{\pm} = \left\{ z \in \mathbb{C}^2;\ 1/2 < |z_1| < 2, \left|\arg z_1 \pm \frac{\pi}{2}\right| > \delta \right\},$$

with $p_0 = (-1, 0)$ and $p_1 = (1, 0)$ and $f(z) = z_2/z_1$. Then the linear functions $\mathscr{L}^m_{p,\Omega^+} f(z)$ and $\mathscr{L}^m_{p,\Omega^-} f(z)$ have different z_2-coefficients, as is easily verified from the fact that in this setting

$$\mathscr{L}^0_{p,\Omega^{\pm}} f(z) = f(p_0) + \sum_{j=1}^{2} (z_j - (p_0)_j) \int_{\gamma^{\pm}} \frac{\partial f}{\partial z_j} (p_0 + \lambda(p_1 - p_0))\, d\lambda,$$

where $\gamma^{\pm}$ are curves from 0 to 1 in the complex plane, such that $p_0 + \lambda(p_1 - p_0) \in \Omega^{\pm}$ when $\lambda \in \gamma^{\pm}$.

Since the mean-value interpolating polynomial has independent meaning in $\mathbb{C}^n$ for entire functions, and since $\mathcal{O}(\mathbb{C}^n)$ is dense in $\mathcal{O}(\Omega)$ whenever Ω is a Runge domain, the linear functional $f \mapsto \mathscr{L}^m_p f(z)$ is densely defined for every such Ω containing p. A natural question which arises in this context is therefore: When is there a continuous extension to all of $\mathcal{O}(\Omega)$? As we have seen, $\mathbb{C}$-convexity is a sufficient condition for such an extension to exist. We now prove that it is also a necessary one.

THEOREM 7.6. *Let k be a fixed positive integer, and m a fixed integer, $0 \leqslant m \leqslant k$. If $\Omega \subset \mathbb{C}^n$ is a Runge domain such that the linear functionals*

$$\mathcal{O}(\mathbb{C}^n) \ni f \mapsto \mathscr{L}^m_p f(z)$$

have continuous extensions to $\mathcal{O}(\Omega)$ for all $p = (p_0, ..., p_k) \subset \Omega$, $z \in \mathbb{C}^n$, then Ω is $\mathbb{C}$-convex.

Proof. Since Ω is Runge its intersection with any complex line consists of simply connected components. Suppose Ω is not $\mathbb{C}$-convex. Then for some complex line ℓ we have that $\Omega \cap \ell$ consists of more than one component. Choose two points, p_0 and p_1, belonging to different components. We may assume that $\ell = \{z_2 = \cdots = z_n = 0\}$ and that $P_0 = (0, 0, ..., 0)$ and $p_1 = (1, 0, ..., 0)$.

Let us first consider the case when $m < k$. In view of Cartan's theorem there are functions $f, g_2, ..., g_n \in \mathcal{O}(\Omega)$ such that

$$f|_{\Omega \cap \ell} \text{ is locally constant}, \qquad f(p_0) = 0, \quad f(p_1) = 1,$$

and

$$\frac{\partial^{m+1} f}{\partial z_1^{m+1}} = z_2 g_2 + \cdots + z_n g_n.$$

Using the fact that Ω is Runge, we can also find sequences $\{f_\nu\}$, $\{g_{2\nu}\}, ..., \{g_{n\nu}\}$ of entire functions such that $f_\nu \to f$ and $g_{j\nu} \to g_j$ in $\mathcal{O}(\Omega)$. Next we put

$$F_\nu = z_2 \frac{\partial^{m+1} f_\nu}{\partial z_1^{m+1}}$$

and

$$G_\nu = z_2(z_2 g_{2\nu} + \cdots + z_n g_{n\nu}),$$

and observe that the sequences $\{F_\nu\}$ and $\{G_\nu\}$ converge in $\mathcal{O}(\Omega)$ to the same limit, namely

$$z_2 \frac{\partial^{m+1} f}{\partial z_1^{m+1}}.$$

Now we wish to calculate the mean-value interpolating polynomials for F_ν and G_ν with respect to the sequence of points $p = (p_0, p_1, p_0, ..., p_0)$, where p_0 occurs k times.

To this end, remember that if ϕ is an entire function, when calculating the coefficients of $\mathscr{L}_p^m \phi(z)$ we can let the γ^j in the definition of the Simplex functional be identity mappings. For example, remembering Definition 7.1, the $z_1^{k-m-1} z_2$-coefficient is given by

$$m! \binom{k}{m} (k-m) \int_0^1 \int_0^{1-\lambda_k} \cdots \int_0^{1-\lambda_k - \cdots - \lambda_2} \times \frac{\partial^{k-m} \phi}{\partial z_1^{k-m-1} \partial z_2} (p_0 + \lambda_1(p_1 - p_0))\, d\lambda_1 \cdots d\lambda_k.$$

Now we apply this to the mean-value interpolating polynomials of F_ν and G_ν. The $z_1^{k-m-1} z_2$-coefficient of $\mathscr{L}_p^m F_\nu$ equals

$$m! \binom{k}{m} (k-m) \int_0^1 \int_0^{1-\lambda_k} \cdots \int_0^{1-\lambda_k - \cdots - \lambda_2} \frac{\partial^k f_\nu}{\partial z_1^k} (\lambda_1, 0, ..., 0)\, d\lambda_1 \cdots d\lambda_k$$
$$= m! \binom{k}{m} (k-m) \left[f_\nu(p_1) - f_\nu(p_0) - \sum_{j=1}^{k-1} \frac{1}{j!} \frac{\partial^j f_\nu}{\partial z_1^j} (p_0) \right].$$

When we let v tend to infinity in this expression, $f_v(p_1) \to 1$ and all the other terms inside the brackets approach zero. So the whole expression tends to $m!\binom{k}{m}(k-m)$.

On the other hand, the $z_1^{k-m-1}z_2$-coefficient of $\mathscr{L}_p^m G_v$ equals

$$m!\binom{k}{m}(k-m)\int_0^1\int_0^{1-\lambda_k}\cdots\int_0^{1-\lambda_k-\cdots-\lambda_2}$$
$$\times\frac{\partial^{k-m}G_v}{\partial z_1^{k-m-1}\,\partial z_2}(\lambda_1, 0, ..., 0)\,d\lambda_1\cdots d\lambda_k = 0.$$

Thus, $\{\mathscr{L}_p^m F_v\}$ and $\{\mathscr{L}_p^m G_v\}$ do not have a common limit, and the theorem is proved in this case.

We now turn to the case when $m = k$. Observe that in this case

$$\mathscr{L}_p^m f = k!\int_{[p_0, ..., p_k]} f.$$

As before we can choose functions $f, h_2, ..., h_n \in \mathcal{O}(\Omega)$ such that

$$f|_{\Omega\cap\ell} \text{ is locally constant,}\qquad f(p_0) = 0,\quad f(p_1) = 1,$$

and

$$\frac{\partial^k f}{\partial z_1^k} = z_2h_2 + \cdots + z_nh_n.$$

Also, we can find sequences $\{f_v\}, \{h_{2v}\}, ..., \{h_{nv}\}$ of entire functions such that $f_v \to f$ and $h_{jv} \to h_j$ in $\mathcal{O}(\Omega)$. Next we construct two sequences of entire functions by putting

$$F_v = \frac{\partial^k f_v}{\partial z_1^k}$$

and

$$H_v = (z_2h_{2v} + \cdots + z_nh_{nv}),$$

and observe that $\{F_v\}$ and $\{H_v\}$ both converge to

$$\frac{\partial^k f}{\partial z_1^k}.$$

However, it is elementary to see that, with $p=(p_0, p_1, p_0, ..., p_0)$ where p_0 occurs k times,

$$\mathscr{L}_p^m F_\nu = k! \left[f_\nu(p_1) - f_\nu(p_0) - \sum_{j=1}^{k-1} \frac{1}{j!} \frac{\partial^j f_\nu}{\partial z_1^j}(p_0) \right],$$

which tends to $k!$ as ν tends to infinity, whereas

$$\mathscr{L}_p^m H_\nu = 0,$$

and so $\{F_\nu\}$ and $\{H_\nu\}$ do not have a common limit. ∎

8. AN INTEGRAL FORMULA FOR THE ERROR

Our aim in the present section is to obtain an integral formula for the error in complex mean-value interpolation, a formula which we will use in the subsequent sections to approximate holomorphic functions.

Our formula will be a generalization of the classical formula for the error in one complex variable Lagrange–Hermite interpolation

$$f(z) - L_p f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \left(\prod_{j=0}^{k} \frac{z-p_j}{\zeta - p_j} \right) \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

For the case $m=0$, i.e., for Kergin interpolation, our error formula will coincide with the one given in [4] and [7].

To obtain the desired error formula, we use the strategy from [4] and [7], viz we use a Fantappiè integral formula to represent our function and then interpolate the kernel, using the continuity and the affine invariance of the mean-value interpolation operator. First we need two auxiliary one variable results.

LEMMA 8.1. *Let n be a positive integer and let $g(t) = 1/t^n$. If $p=(p_0, ..., p_k)$ is a sequence of points from $\mathbb{C}\backslash\{0\}$, then*

$$[p_0, ..., p_k]\, g = \frac{(-1)^k}{p_0 p_1 \cdots p_k} \sum_{|\alpha| = n-1} \frac{1}{p^\alpha},$$

where, using ordinary multi-index notation, $p^\alpha = p_0^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}$.

Proof. See Lemma 1.10 in [7]. ∎

LEMMA 8.2. *Let* $g(t)=1/t^n$ *and let* $p=(p_0, ..., p_k)$ *be a sequence of points from* $\mathbb{C}\backslash\{0\}$. *If* $0 \leqslant m \leqslant \min\{k, n-1\}$, *then*

$$g(t)-L_p^m g(t)=\sum_{j=0}^{m} \sum_{\substack{p' \subset p \\ \#p'=k+1-j}} C_{m,n}\left(\prod_{p_r \in p'} (t-p_r)\right)$$
$$\times \frac{(-1)^{k+1-j}}{t^{m-j+1} p_0 p_1 \cdots p_k} \sum_{\substack{|\alpha|+\beta= \\ n-1-m}} \binom{m-j+\beta}{\beta} \frac{1}{p^\alpha t^\beta},$$

where $C_{m,n}=m!(n-m-1)!/(n-1)!$.

Proof. First we recall the formula introduced in the proof of Theorem 3.2. That is, with $D^{-m}g$ being any function such that $D^m(D^{-m}g)=g$ and L_p the ordinary Lagrange operator,

$$L_p^m g=D^m(L_p(D^{-m}g)).$$

From this formula we immediately get

$$g-L_p^m g=D^m(I-L_p)\, D^{-m}g,$$

where I is the identity operator. For $g(t)=t^{-n}$ we can take

$$D^{-m}g(t)=(-1)^m \frac{(n-m-1)!}{(n-1)!}\, t^{-n+m}.$$

It is well known that the error in Lagrange interpolation of a function h at the points p is given by

$$(I-L_p)\, h(t)=(t-p_0)(t-p_1)\cdots(t-p_k)[p_0, ..., p_k, t]\, h.$$

Plugging into this formula our explicitly calculated expression for $D^{-m}g$, using Lemma 8.1 to calculate the divided difference, and finally differentiating m times yields the desired formula. ▮

Now we can give an integral formula for the error in complex mean-value interpolation.

THEOREM 8.3. *Let* Ω *be a bounded* $\mathbb{C}$*-convex domain in* $\mathbb{C}^n$ *with* C^2 *boundary and with defining function* ρ, *i.e.,* $\Omega=\{\rho(z)<0\}$. *Let* f *be a function holomorphic in* Ω *and continuous up to the boundary, and let* $p=(p_0, ..., p_k)$ *be a sequence of points from* Ω. *Then the following formula holds for* $0 \leqslant m \leqslant \min\{k, n-1\}$:

$$
\begin{aligned}
&f(z)-\mathscr{L}^m_{p,\Omega}f(z)\\
&\quad=\frac{1}{(2\pi i)^n}\int_{\partial\Omega}\frac{m!(n-m-1)!}{(n-1)!}\sum_{j=0}^{m}\sum_{\substack{p'\subset p\\ \#p'=k+1-j}}\left(\frac{\prod_{p_r\in p'}\langle\rho'(\zeta),z-p_r\rangle}{\prod_{r=0}^{k}\langle\rho'(\zeta),\zeta-p_r\rangle}\right)\\
&\qquad\times(-1)^{k+1-j}\sum_{\substack{|\alpha|+\beta=\\ n-1-m}}\binom{m-j+\beta}{\beta}\\
&\qquad\times\frac{f(\zeta)\,\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}}{\langle\rho'(\zeta),\zeta-p\rangle^{\alpha}\,\langle\rho'(\zeta),\zeta-z\rangle^{\beta+m-j+1}}.
\end{aligned}
$$

Proof. Since Ω is $\mathbb{C}$-convex, it is also lineally convex. Hence every complex tangent plane $T_\zeta=\{z\in\mathbb{C}^n;\ \langle\rho'(\zeta),\zeta-z\rangle=0\}$ lies entirely outside Ω. Therefore the mapping $\partial\Omega\times\Omega\ni(\zeta,z)\mapsto\langle\rho'(\zeta),\zeta-z\rangle$ is non-vanishing and hence it gives rise to a Fantappiè formula (cf. [1])

$$
f(z)=\frac{1}{(2\pi i)^n}\int_{\partial\Omega}\frac{f(\zeta)\,\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}}{\langle\rho'(\zeta),\zeta-z\rangle^n}.
$$

The idea is now to interpolate the kernel function $z\mapsto\langle\rho',\zeta-z\rangle^{-n}$ in order to obtain the desired formula for $f-\mathscr{L}^m_{p,\Omega}f$.

In view of the representation formula above and the continuity of the mean-value interpolation operator we have the formula

$$
\begin{aligned}
&f(z)-\mathscr{L}^m_{p,\Omega}f(z)\\
&\quad=\frac{1}{(2\pi i)^n}\int_{\partial\Omega}f(\zeta)\left(\frac{1}{\langle\rho'(\zeta),\zeta-z\rangle^n}-\mathscr{L}^m_{p,\Omega}\frac{1}{\langle\rho'(\zeta),\zeta-z\rangle^n}\right)\\
&\qquad\times\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}.
\end{aligned}
$$

Combining this with Lemma 8.2 and the affine invariance of the mean-value interpolation operator proved in Theorem 7.3(iv) yields the result. ∎

Remark 8.4. For $m=0$ we recover the error formula for complex Kergin interpolation given by Andersson and Passare in [4].

Remark 8.5. In Theorem 8.3 the restriction $m\leqslant n-1$ is imposed. The reason for this is of a practical nature and evident in the proof of Lemma 2.8: for $m\geqslant n$ logarithms are introduced when taking an mth primitive function of $1/t^n$, and there is no longer a convenient explicit formula (like the one in Lemma 8.1) for the divided difference. We point out that, in spite of the restriction, Theorem 8.3 contains all of the special instances of mean-value interpolation that have been separately studied in the multivariate real case, viz. $m=0$, $m=1$, and $m=n-1$.

9. APPROXIMATION OF ENTIRE FUNCTIONS

As an application of our integral formula for the error in mean-value interpolation, we now generalize results of Andersson and Passare [4] and Bloom [7] concerning approximation of entire functions. The following definition was made in [4].

DEFINITION 9.1. Let v be a norm on $\mathbb{C}^n$ and λ a positive real number.

(i) Let f be an entire function on $\mathbb{C}^n$. The λ-type of f with respect to v is given by

$$\tau_v(\lambda) = \limsup_{r \to \infty} \frac{\log M_v(r)}{r^\lambda},$$

where $M_v(r)$ is the maximum of $|f|$ in the closed ball $v(z) \leqslant r$.

(ii) Let $p = (p_0, p_1, ...)$ be a sequence of points from $\mathbb{C}^n$. The λ-density of p with respect to v is given by

$$\delta_v(\lambda) = \liminf_{r \to \infty} \frac{N_v(r)}{r^\lambda},$$

where $N_v(r)$ is the number of points from p in the closed ball $v(z) \leqslant r$.

The theorem we will give below, about the convergence of the mean-value interpolating polynomials to entire functions, was stated and proved in [4] for the case $m = 0$, i.e., for Kergin interpolation. The same type of result (again for Kergin interpolation) was given already in [7] and [8], although in a less general setting. We use methods similar to those of [4, 7 and 8] to prove our convergence theorem for general mean-value interpolation.

THEOREM 9.2. *Let f be an entire function on $\mathbb{C}^n$ and $p = (p_0, p_1, ...)$ a discrete sequence of points from $\mathbb{C}^n$ with $v(p_j) \leqslant v(p_{j+1})$ for all j. Let m be an integer such that $0 \leqslant m \leqslant n - 1$.*

For any complex-homogenous norm v on $\mathbb{C}^n$ and any positive real number λ, let $\tau_v(\lambda)$ be the λ-type of f and $\delta_v(\lambda)$ the λ-density of p, both measured with the norm v. If the inequality

$$\frac{\tau_v(\lambda)}{\delta_v(\lambda)} < c(\lambda) := \int_0^{1/2} \frac{t^{\lambda - 1}\, dt}{1 - t}$$

holds, then the mean-value interpolating polynomials $\mathscr{L}^m_{p^k} f$ of f with respect to $p^k = (p_0, ..., p_k)$ converge to f uniformly on compact sets of $\mathbb{C}^n$.

Moreover, the constant $c(\lambda)$ is the largest one with this property.

Proof. To being with, we assume that v is smooth away from the origin. We want to investigate the error (for z in some compact set) after interpolation at the points $p^k=(p_0, ..., p_k)$. Put $v(p_j)=r_j$ and assume that $v(z)\leqslant r<r_k<R$. Then, by Theorem 8.3, this error is

$$\begin{aligned}
&f(z)-\mathscr{L}^m_{p,\Omega}f(z)\\
&\quad=\frac{1}{(2\pi i)^n}\int_{v(\zeta)=R} C_{m,n}\sum_{j=0}^{m}\sum_{\substack{p'\subset p\\ \#p'=k+1-j}}\left(\frac{\prod_{p_r\in p'}\langle v'(\zeta), z-p_r\rangle}{\prod_{r=0}^{k}\langle v'(\zeta), \zeta-p_r\rangle}\right)\\
&\qquad\times(-1)^{k+1-j}\sum_{\substack{|\alpha|+\beta=\\ n-1-m}}\binom{m-j+\beta}{\beta}\\
&\qquad\times\frac{f(\zeta)\,\partial v\wedge(\bar\partial\partial v)^{n-1}}{\langle v'(\zeta), \zeta-p\rangle^{\alpha}\langle v'(\zeta), \zeta-z\rangle^{\beta+m-j+1}},
\end{aligned}$$

where $C_{m,n}=m!(n-m-1)!/(n-1)!$.

We now turn to estimating this expression. To this end we first observe that

$$C_{m,n}\binom{m-j+\beta}{\beta}\leqslant 1.$$

Further, as in [4] we can prove that, for each j,

$$|\langle v'(\zeta), z-p_j\rangle|\leqslant \tfrac{1}{2}(r+r_j)$$

and

$$|\langle v'(\zeta), \zeta-p_j\rangle|\geqslant \tfrac{1}{2}(R-r_j).$$

We also observe that $i^{-n}\partial v\wedge(\bar\partial\partial v)^{n-1}$ is a positive measure on $\{z;\ v(z)=R\}$ with mass $(\pi R)^n$.

Using these facts and making the obvious estimates in the above formula for the error, much in the same fashion as in [4], we get that

$$\begin{aligned}
&|f(z)-\mathscr{L}^m_{p^k}f(z)|\\
&\quad\leqslant M_v(R)(m+1)\binom{k+n-m}{n-m-1}\binom{k+1}{m}\left(\prod_{j=m}^{k}\frac{r+r_j}{R-r_j}\right)\left(1-\frac{r_k}{R}\right)^{-n}.
\end{aligned}$$

We will show that there is a sequence $R = R(k)$ such that, as $k \to \infty$, the logarithm of the right-hand side tends to $-\infty$.

Using the easily verified relation

$$\sum_{j=0}^{k} \log\left(\frac{r+r_j}{R-r_j}\right) = (k+1)\log\left(\frac{r+r_k}{R-r_k}\right) - (r+R)\int_0^{r_k} \frac{N_\nu(t)\,dt}{(r+t)(R-t)},$$

we thus have to examine the following expression:

$$\begin{aligned}
&\log M_\nu(R) + \log(m+1) + \log\left(\binom{k+1}{m}\binom{k+n-m}{n-m-1}\right)\\
&+ (k+1)\log\left(\frac{r+r_k}{R-r_k}\right) - (r+R)\int_0^{r_k} \frac{N_\nu(t)\,dt}{(r+t)(R-t)}\\
&+ \sum_{j=0}^{m-1} \log\left(\frac{R-r_j}{r+r_j}\right) + \log\left(1-\frac{r_k}{R}\right)^{-n}.
\end{aligned}$$

Now we choose $R = R(k) = 2r + 2r_k$. Then we get:

(1) The second and seventh terms are of no consequence. Their sum is bounded by some constant C_1.

(2) The third term can be estimated by $C_2 + C_3 \log k$, for some constants C_2 and C_3, since for large k

$$\begin{aligned}
\binom{k+1}{m}\binom{k+n-m}{n-m-1} &\leqslant \frac{(k+1)!\,(k+n-m)!}{(k+1-m)!\,(k+1)!}\\
&\leqslant (2k)^m\,(2k)^{n-m-1} \leqslant (2k)^{n-1}.
\end{aligned}$$

(3) To deal with the fourth term, we note that for $x > 0$ and small, $\log(1-x) < -x$. Hence

$$(k+1)\log\left(1-\frac{r}{2r+r_k}\right) < -(k+1)\frac{r}{2r+r_k}.$$

(4) As for the sixth term, it is not greater than $C_4 + C_5 \log r_k$, for some constants C_4 and C_5.

(5) The crucial terms are the first and the fifth. This is where the type and density are used. We have that

$$\log M_\nu(R) - (r+R) \int_0^{r_k} \frac{N_\nu(t)\, dt}{(r+t)(R-t)}$$

$$= R^\lambda \left(\frac{\log M_\nu(R)}{R^\lambda} - \int_0^{r_k} \frac{(r+R)\,(N_\nu(t)/R^\lambda)\, dt}{(r+t)(R-t)} \right)$$

$$= R^\lambda \left(\frac{\log M_\nu(R)}{R^\lambda} - \int_0^{r_k/R} \frac{(r+R)\,(N_\nu(sR)/(sR)^\lambda)\, s^\lambda\, ds}{(r+sR)(1-s)} \right)$$

$$\leqslant -C_6 R^\lambda \leqslant -C_6 r_k^\lambda,$$

for some constant $C_6 > 0$, since for large enough R the expression inside the parentheses is $\leqslant \tau_\nu(\lambda) - \delta_\nu(\lambda)\, c(\lambda)$.

Bringing it all back home, we end up with

$$\log |f(z) - \mathscr{L}^m_{p^k} f(z)|$$
$$\leqslant C_1 + C_2 + C_4 + C_3 \log k + C_5 \log r_k - \frac{r(k+1)}{2r + r_k} - C_6 r_k^\lambda,$$

which is easily seen to tend to $-\infty$ as $k \to \infty$. The theorem is proved in the smooth case.

If ν is not smooth we proceed as in [4]. For each $\varepsilon > 0$ we can find a smooth norm $\tilde{\nu}$ such that $(1-\varepsilon)\, \nu < \tilde{\nu} < \nu$. It is then clear that

$$\frac{\tau_\nu(\lambda)}{(1-\varepsilon)^\lambda} \geqslant \tau_{\tilde{\nu}}(\lambda),$$

$$\delta_{\tilde{\nu}}(\lambda) \geqslant \delta_\nu(\lambda),$$

and so we still have that

$$\frac{\tau_{\tilde{\nu}}(\lambda)}{\delta_{\tilde{\nu}(\lambda)}} < c(\lambda)$$

if ε is chosen small enough. Now the smooth result applies.

That the constant $c(\lambda)$ is the largest one possible is proved in [4] for the case $m = 0$ and their argument actually goes through for any m such that $0 \leqslant m \leqslant n-1$. ∎

Remark 9.3. The restriction $m \leqslant n-1$, explained in Remark 8.5, can be dropped when $n = 1$. Indeed, this is a consequence of the formula $L_p^m f = D^m L_p D^{-m} f$ and the fact that the theorem holds for Lagrange–Hermite interpolation.

10. APPROXIMATION OF HOLOMORPHIC FUNCTIONS

In this section we shall generalize results of Bloom and Calvi concerning approximation of functions holomorphic in a bounded domain. In [9] they consider Kergin interpolation at a triangular array of points $\{p_{kj}\}$ in a compact set K in $\mathbb{C}^n$ and investigate the problem of finding a domain Ω as small as possible such that for every f holomorphic in Ω the Kergin polynomial at $(p_{k0}, p_{k1}, ..., p_{kk})$ exists and converges to f uniformly on K as k tends to ∞. They give conditions on the distribution of points and on the domain Ω which ensures this. As we shall see, their conditions work for general mean-value interpolation as well.

First we introduce some notation and now we are following [9] quite closely. For a compact subset $K \subset \mathbb{C}^n$ we write $\mu \in \mathscr{M}(K)$ if μ is a positive Borel measure supported by K. For each nonzero linear form A on $\mathbb{C}^n$, μ^A denotes the plane measure in $\mathscr{M}(A(K))$ defined by

$$\mu^A(f) = \int_K (f \circ A)\, d\mu.$$

The negative of its logarithmic potential is given by

$$\Psi_\mu(A, u) = \mu^A(\log |u - \cdot|) = \int_K \log |u - A(z)|\, d\mu(z),$$

which, as a function of u, is subharmonic on $\mathbb{C}$.

We also set

$$M_\mu(A) = \sup \{\Psi_\mu(A, u);\ u \in A(K)\}.$$

Note that since $\Psi_\mu(A, \cdot)$ is upper semicontinuous, $M_\mu(A)$ is attained at some point of $A(K)$.

We further let $F_\mu(A)$ be the plane compact set defined by

$$F_\mu(A) := \{u \in \mathbb{C};\ \Psi_\mu(A, u) \leqslant M_\mu(A)\}.$$

Finally, given an array of points $\{p_{kj};\ j = 0, 1, ..., k, k = 0, 1, 2, ...\}$ we write δ_{kj} for the Dirac measure at the point p_{kj}. And throughout this section p_k will denote the kth stage of the array, i.e., p_k is the sequence $(p_{k0}, p_{k1}, ..., p_{kk})$.

THEOREM 10.1. *Let $\Omega = \{\rho < 0\}$ be a bounded $\mathbb{C}$-convex domain with ρ a C^2 defining function. Let $P = \{p_{kj};\ j = 0, ..., k, k = 0, 1, 2, ...\}$ be an array of points in a compact set $K \subset \Omega$ such that the sequence of measures*

$$\mu_k := \frac{1}{k+1} \sum_{j=0}^{k} \delta_{kj}, \qquad k = 0, 1, 2, ...,$$

weakly converges to some measure μ in $\mathscr{M}(K)$. If $0 \leqslant m \leqslant n-1$ and if for every nonzero linear form A we have that

$$A(\Omega) \supset F_\mu(A),$$

then for every function f holomorphic in a neighborhood of $\bar{\Omega}$

$$\lim_{k \to \infty} \sup_K |f - \mathscr{L}^m_{p_k, \Omega} f| = 0.$$

Proof. In view of the error formula and the fact that for ζ on the boundary of Ω, $\langle \rho'(\zeta), \zeta - z \rangle$ does not vanish for $z \in \Omega$, it is enough to prove that

$$\lim_{k \to \infty} \sum_{j=0}^{m} \sum_{\substack{p' \subset p_k \\ \# p' = k+1-j}} \left(\frac{\prod_{p_r \in p'} \langle \rho'(\zeta), z - p_r \rangle}{\prod_{r=0}^{k} \langle \rho'(\zeta), \zeta - p_r \rangle} \right) = 0.$$

This poses no problem, since in [9] it is proved that, under the assumptions of the theorem, for some constant $c < 1$,

$$\left| \prod_{j=0}^{k} \frac{\langle \rho'(\zeta), z - p_{kj} \rangle}{\langle \rho'(\zeta), \zeta - p_{kj} \rangle} \right| \leqslant c^{k+1},$$

for all $z \in K$, and all $\zeta \in \partial\Omega$. ∎

In [9] there are also examples of so-called extremal arrays for Kergin interpolation. An array P is extremal for a compact K if $P \subset \partial K$ and for each function f holomorphic in a neighborhood of K the interpolating polynomials exist at each stage of the array and converge to f. The examples given in [9] for Kergin interpolation generalize immediately to mean-value interpolation.

Let $K \subset \mathbb{C}^n$ be a compact circular set of center 0, i.e., let K satisfy the condition

$$z \in K, \lambda \in \mathbb{C}, \qquad |\lambda| \leqslant 1 \Rightarrow \lambda z \in K.$$

A positive Borel measure μ on ∂K is said to be invariant if for every $\theta \in \mathbb{R}$ and every continuous function f on ∂K

$$\int f(e^{i\theta} t)\, d\mu(t) = \int f(t)\, d\mu(t).$$

Now we have the following.

THEOREM 10.2. *Let K be a circular compact convex set of center 0 in $\mathbb{C}^n$. If $P = \{ p_{kj};\ j = 0, ..., k, k = 0, 1, 2, ...\}$ is an array of points on ∂K such that*

$$\frac{1}{k+1} \sum_{j=0}^{k} \delta_{kj} \xrightarrow{*} d\mu,$$

where $d\mu$ is an invariant probability measure on ∂K and δ_{kj} is the Dirac measure at the point p_{kj}, then P is an extremal array for mean-value interpolation, i.e., for every function f holomorphic in a neighborhood of K,

$$\lim_{k \to \infty} \sup_K |f - \mathscr{L}^m_{p_k, \Omega} f| = 0.$$

Proof. The argument given in [9] for the case $m = 0$ actually goes through for any m, in view of Theorem 10.1 above. ∎

Remark 10.3. Note that a $\mathbb{C}$-convex circular set is automatically convex. In [6, Example 2.2.3] this is proved for Reinhard sets, but the proof is easily adapted to circular sets.

From Theorem 10.2 it is easy to construct explicit examples of extremal arrays for mean-value interpolation. All examples of extremal arrays for Kergin interpolation obtained in [9] are valid also for general mean-value interpolation. Here we just cite the simplest example and refer the reader to [9] for extremal arrays in the polydisc, for example.

EXAMPLE 10.4. Take a point $a \in \partial K$ and define an array $\{p_{kj}\}$ by

$$p_{kj} = a \exp\left(\frac{2i\pi j}{k+1}\right), \qquad j = 0, ..., k.$$

The corresponding sequence of measures converges to the usual measure on the circle in ∂K of center 0 passing through a, and so the array is extremal.

11. MEAN-VALUE INTERPOLATION AND THE FANTAPPIÈ TRANSFORM

The notion of $\mathbb{C}$-convexity, crucial for complex mean-value interpolation, is closely connected to the Fantappiè transform. Exploiting this connection gives rise to a different way of viewing mean-value interpolation. This was done in [5] for Kergin interpolation and in this section we shall generalize the results of [5] to mean-value interpolation.

First we need a definition.

DEFINITION 11.1. Let Ω be a subset of $\mathbb{C}^n$ such that $0 \in \Omega$. The dual complement of Ω is defined to be

$$\Omega^* = \{\zeta \in (\mathbb{C}^n)^*;\ 1 + \langle z, \zeta \rangle \neq 0, \forall z \in \Omega\}.$$

Let Ω be a domain in $\mathbb{C}^n$ with $0 \in \Omega$. If μ is an analytic functional on $\mathcal{O}(\Omega^*)$, i.e., $\mu \in \mathcal{O}'(\Omega^*)$, then the Fantappiè transform of μ is

$$\mathscr{F}\mu(z) = \mu_\zeta \left(\frac{1}{1 + \langle z, \zeta \rangle} \right).$$

Since μ is representable by some measure, $\tilde{\mu}$, this could also be written

$$\mathscr{F}\mu(z) = \int_{\Omega^*} \frac{d\tilde{\mu}(\zeta)}{1 + \langle z, \zeta \rangle}.$$

The Fantappiè transformation is a continuous linear mapping

$$\mathscr{F} : \mathcal{O}'(\Omega^*) \to \mathcal{O}(\Omega).$$

We refer the reader to [2, 5 and 23] for more about the Fantappiè transform. The following theorem is fundamental.

THEOREM 11.2. *Let Ω be a domain in $\mathbb{C}^n$. Then Ω is $\mathbb{C}$-convex if and only if $\mathscr{F}$ is a topological isomorphism.*

Proof. See [2] and [23]. ∎

One part of this theorem states that if Ω is $\mathbb{C}$-convex, then to each $f \in \mathcal{O}(\Omega)$ there corresponds a unique $\mu \in \mathcal{O}'(\Omega^*)$ such that $f = \mathscr{F}\mu$. What this in essence is saying is that every function holomorphic in a $\mathbb{C}$-convex domain can be thought of as a superposition of one-variable functions.

Using Theorem 11.2 and postulating affine invariance lead to a different proof of the existence and uniqueness of the complex mean-value interpolation operator. What we do is simply to define the interpolating polynomial in terms of the inverse Fantappiè transform of the function. This approach was applied to Kergin interpolation in [5]. For general mean-value interpolation, with this approach, Theorem 7.3 can be replaced by the following.

THEOREM 11.3. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain, $p = (p_0, ..., p_k)$ a sequence of points in Ω, and m an integer, $0 \leqslant m \leqslant k$. Then to each function $f \in \mathcal{O}(\Omega)$ there is a unique polynomial $\mathcal{L}^m_{p,\Omega} f$ of degree at most $k-m$, such that it is invariant under complex affine mappings, depends linearly and continuously on f, and*

$$\int_{[p']} (f - \mathcal{L}^m_{p,\Omega} f) = 0,$$

for all subsequences $p' \subset p$ such that $\# p' = m+1$.

Moreover,

(i) *it is independent of the ordering of the points,*

(ii) *it is holomorphic as a function of p,*

(iii) *it is a projection onto $\Pi_{k-m}(\mathbb{C}^n)$,*

(iv) *it is invariant under restriction,*

(v) *it reduces to a Taylor polynomial if p consists of one point repeated $k+1$ times, and*

(vi) *it is associative: $p \subset q \Rightarrow \mathcal{L}^m_{p,\Omega} \mathcal{L}^m_{q,\Omega} f = \mathcal{L}^m_{p,\Omega} f$.*

Proof. First let us establish the uniqueness. To this end we note that if f is an entire function of the form $f(z) = (g \circ A)(z)$, where g is a one-variable function and $A(z) = \langle z, a \rangle$ for some point $a \in \mathbb{C}^n$, then it is clear from the postulated affine invariance that

$$\mathcal{L}^m_{p,\Omega} f(z) = \mathcal{L}^m_{p,\Omega} (g \circ A)(z) = ((L^m_{A(p), A(\Omega)} g) \circ A)(z).$$

Here $L^m_{A(p), A(\Omega)} g$ is the *unique* one-variable polynomial of degree $k-m$ such that

$$\int_{[A(p')]} (g - L^m_{A(p), A(\Omega)} g) = 0,$$

for all subsequences $p' \subset p$ such that $\# p' = m+1$ (cf. Remark 3.4). Hence $\mathcal{L}^m_{p,\Omega} f$ is uniquely determined for any entire function f of the above form, in particular for polynomials of that type. But by polarization every polynomial is a finite sum of such polynomials. Thus, since $\mathcal{L}^m_{p,\Omega} f$ depends linearly and continuously on f and any $\mathbb{C}$-convex domain is Runge, we obtain the desired uniqueness.

To establish the existence of $\mathscr{L}^m_{p,\Omega} f$ we make a slight change of notation. First we put $h(t)=1/(1+t)$ and then for any point sequence $q=(q_1, ..., q_k)$ we let $H(q_1, ..., q_k, t)$ be the unique polynomial in t of degree at most $k-m$ such that

$$\int_{[q']} (h-H)=0,$$

for all subsequences $q' \subset q$ such that $\# q' = m+1$ (cf. Remark 3.4).

Next, we let $\mu \in \mathcal{O}'(\Omega^*)$ be the inverse Fantappiè transform of f, i.e.,

$$f(z)=\mu_\zeta\left(\frac{1}{1+\langle z, \zeta\rangle}\right).$$

This representation is possible by Theorem 11.2. Now we define the mean value interpolating polynomial by putting

$$\mathscr{L}^m_{p,\Omega} f(z)=\mu(H(\langle p_0, \cdot\rangle, ..., \langle p_k, \cdot\rangle, \langle z, \cdot\rangle)).$$

It is clear that this definition gives continuity with respect to f. And representing μ by the measure $\tilde{\mu}$ as above, we see that

$$\begin{aligned}\int_{[p_0, ..., p_{j+m}]} & (f-\mathscr{L}^m_{p,\Omega} f) \\ &=\int_{[p_0, ..., p_{j+m}]} \int_{\Omega^*} \left(\frac{1}{1+\langle \zeta, z\rangle} - H(\langle p_0, \zeta\rangle, ..., \langle z, \zeta\rangle)\right) d\tilde{\mu}(\zeta),\end{aligned}$$

which vanishes for $j=0, ..., k-m$ by the choice of H, since changing the order of integration is justifiable.

Also, for any affine mapping T, we have

$$\begin{aligned}((\mathscr{L}^m_{T(p), T(\Omega)} f)\circ T)(z) &= \mu(H(\langle T(p_0), \cdot\rangle, ..., \langle T(p_k), \cdot\rangle, \langle T(z), \cdot\rangle)) \\ &= \mu(H(\langle p_0, T^* \cdot\rangle, ..., \langle p_k, T^*\cdot\rangle, \langle z, T^*\cdot\rangle)),\end{aligned}$$

where T^* is the adjoint mapping. On the other hand, since

$$(f\circ T)(z)=\mu\left(\frac{1}{1+\langle T(z), \cdot\rangle}\right)=\mu\left(\frac{1}{1+\langle z, T^*\cdot\rangle}\right),$$

it is also clear that

$$(\mathscr{L}^m_{p,\Omega}(f\circ T))(z)=\mu(H(\langle p_0, T^*\cdot\rangle, ..., \langle p_k, T^*\cdot\rangle, \langle z, T^*\cdot\rangle)),$$

and so the affine invariance does indeed hold. Now the other properties follow from Theorem 7.3, but these properties can of course easily be proved directly. The property (i) is obvious, (ii) is clear from the definition, (iv) follows as in Theorem 7.3, and for the other properties one observes that they hold in one variable, and hence, via polarization, for polynomials, which are dense in $\mathcal{O}(\Omega)$. ▮

12. NUMERICAL ANALYSIS APPROACH

In [21] Waldron gives error formulae for mean-value interpolation in $\mathbb{R}^n$. These formulae are then used to prove uniform estimates. The results obtained imply that a numerical scheme based on mean-value interpolation has the highest possible order. This approach can of course also be applied to complex mean-value interpolation.

In fact, the formulae of [21] are easily seen to hold for entire functions, and since the entire functions are dense in the holomorphic functions in any $\mathbb{C}$-convex domain, the extensions are almost immediate.

THEOREM 12.1. *Let Ω be a $\mathbb{C}$-convex domain and $p=(p_0, ..., p_k)$ a sequence of points in Ω. If m is an integer, $0 \leqslant m \leqslant k$, then for any function $f \in \mathcal{O}(\Omega)$*

$$f(z) - \mathcal{L}^m_{p,\Omega} f(z) = m! \sum_{j=0}^{m} \sum_{\substack{p' \subset p \\ \#p' = k+1-j}} \int_{[z, ..., z, p]} D_{z-p'} f,$$

where in $[z, ..., z, p]$ the variable z occurs $m-j+1$ times.

Proof. This formula is proved in [21] for the real case, and by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and taking the real and imaginary part separately, it also holds for entire functions. Since Ω is $\mathbb{C}$-convex, $\mathcal{O}(\mathbb{C}^n)$ is dense in $\mathcal{O}(\Omega)$, and so we can find a sequence f_ν of entire functions converging to f uniformly on compact subsets of Ω and the result follows by the continuity of the mean-value interpolation operator. ▮

This error formula contains derivatives of orders $k+1-m, ..., k+1$. Since the degree of the interpolating polynomial space is $k-m$, it is, from the numerical analysis point of view, desirable to have an error formula that contains only derivatives of order $k-m+1$. This is done for the real case in [21], where actually a stronger theorem about the derivatives of the error is proved [21, Theorem 5.12]. The complex version of that result is the following.

THEOREM 12.2. *Let Ω be a $\mathbb{C}$-convex domain and $p=(p_0, ..., p_k)$ a sequence of points in Ω. If m is any integer such that $0 \leqslant m \leqslant k$ and $|\alpha| = l \leqslant k-m$, then for any function $f \in \mathcal{O}(\Omega)$*

$$D^\alpha(f(z) - \mathcal{L}^m_{p,\Omega} f(z)) = (m+l)! \sum_{j=k-m-l}^{k} \sum_{\substack{p' \subset p^{j-1} \\ \# p' = m+j+l-k}} \int_{[z, ..., z, p^j]} D_{z-p'} D_{z-p_j} D^\alpha f,$$

where in the expression $[z, ..., z, p^j]$, the variable z occurs $k+1-j$ times.

Proof. As in the proof of Theorem 12.1 the result follows by continuity from the same result in the real case (see [21, Theorem 5.12]). ∎

Remark 12.3. Taking $l=0$ in the above formula gives the following formula for the error in complex mean-value interpolation:

$$f(z) - \mathcal{L}^m_{p,\Omega} f(z) = m! \sum_{j=k-m}^{k} \sum_{\substack{p' \subset p^{j-1} \\ \# p' = m+j-k}} \int_{[z, ..., z, p^j]} D_{z-p'} D_{z-p_j} f,$$

where in the expression $[z, ..., z, p^j]$ the variable z occurs $k+1-j$ times. This formula contains only derivatives of f of order $k-m+1$.

From Theorem 12.1 and Theorem 12.2 one can obtain uniform estimates for the error. In the real case, this is done in [21]. Following the approach of [21] we introduce the following notation. Given a sequence of points $p=(p_0, ..., p_k)$, let

$$h_{z,p} := \max_{0 \leqslant j \leqslant k} |z - p_j|.$$

To measure the size of the jth derivative of f at $z \in \mathbb{C}^n$, we use the seminorm

$$\|D^j f\|(z) := \sup_{\substack{w_1, ..., w_j \in \mathbb{C}^n \\ |w_i| \leqslant 1}} |D_{w_1} \cdots D_{w_j} f(z)|.$$

To measure the size of the jth derivative of f over some compact set $K \subset \mathbb{C}^n$, we use the seminorm

$$\||f|\|_{j,K} := \sup_{z \in K} \|D^j f\|(z).$$

From Theorem 12.1 we obtain the following uniform estimate for the error.

THEOREM 12.4. *Let Ω be a $\mathbb{C}$-convex domain, $p=(p_0, ..., p_k)$ a sequence of points from Ω, m an integer such that $0 \leqslant m \leqslant k$, and $f \in \mathcal{O}(\Omega)$. Then for any compact set $K \subset \Omega$ that contains the points p, the following estimate holds for $z \in K$:*

$$|f(z) - \mathcal{L}^m_{p,\Omega} f(z)| \leqslant \sum_{j=0}^{m} C^{m,j}_{p,K} (h_{z,p})^{k-j} \, |||f|||_{k-j,K},$$

for some constants $C^{m,j}_{p,K}$.

Proof. The proof is immediate from Theorem 12.1 (cf. Proposition 6.1 in [21]), using the continuity of the mean-value interpolation operator and the density in $\mathcal{O}(\Omega)$ of $\mathcal{O}(\mathbb{C}^n)$. ▮

Remark 12.5. The constants $C^{m,j}_{p,K}$ introduced in this estimate essentially depend on the measure of the deformed simplex over which we integrate in the formula of Theorem 12.1. If f is an entire function, we can integrate over the usual simplex, so we can take

$$C^{m,j}_{p,K} = \frac{m!}{(k+m-j)!} \binom{k+1}{j}$$

in this case. The same is true if Ω is convex or if f can be continued to a function holomorphic on the convex hull of the points p.

Theorem 12.2 gives rise to the following estimates.

THEOREM 12.6. *Let Ω be a $\mathbb{C}$-convex domain, $p=(p_0, ..., p_k)$ a sequence of points from Ω, m an integer such that $0 \leqslant m \leqslant k$, and $|\alpha| = l \leqslant k-m$. Then for any $f \in \mathcal{O}(\Omega)$ and any compact set $K \subset \Omega$ that contains the points p, the following estimate holds for $z \in K$:*

$$|D^\alpha(f(z) - \mathcal{L}^m_{p,\Omega} f(z))| \leqslant C^{m,l}_{p,K} (h_{z,p})^{k-m-l} \, |||f|||_{k-m+1,K},$$

for some constant $C^{m,l}_{p,K}$.

Proof. The real result is given in [21, Theorem 6.2]. Our result follows in much the same way, using Theorem 12.2, the continuity of the mean-value interpolation operator, and the density of $\mathcal{O}(\mathbb{C}^n)$ in $\mathcal{O}(\Omega)$. ▮

Remark 12.7. Again the constant $C^{m,l}_{p,K}$ is essentially depending on the measure of the deformed simplex over which we integrate in the formula of Theorem 12.2. If f is an entire function we can take

$$C^{m,l}_{p,K} = \frac{1}{(k-m-l)!}.$$

The same constant applies if Ω is convex or if f can be continued to a function holomorphic on the convex hull of the points p.

As was pointed out in [21], results such as Theorem 12.2 and Theorem 12.6 are precisely what numerical analysts want to ensure that their scheme (e.g., a $\mathscr{L}^m_{p,\Omega}$ finite element) has the maximum possible order.

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